The Haemers Bound of Noncommutative Graphs

Sander Gribling and **Yinan Li** (CWI & QuSoft)

January 10, 2020

QIP 2020, Shenzhen. Preprint in preparation.

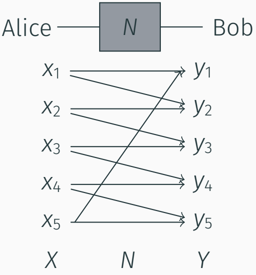
What is Haemers bound?

What is a noncommutative graph?

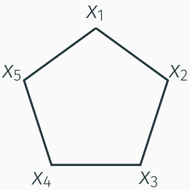
Why should we care about it?

Zero-error Communication through Classical Channels

Classical communication

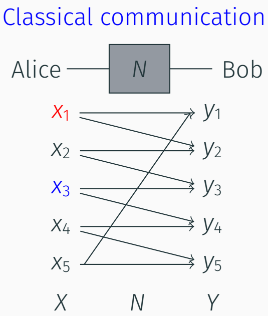


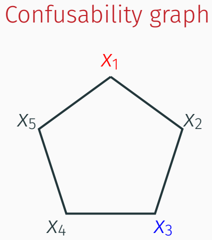
Confusability graph



$x_i \sim x_j$ if $\exists y \in Y$, s.t. $N(y|x_i)N(y|x_j) > 0$.

Zero-error Communication through Classical Channels



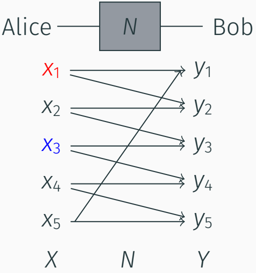


$x_i \sim x_j$ if $\exists y \in Y$, s.t. $N(y|x_i)N(y|x_j) > 0$.

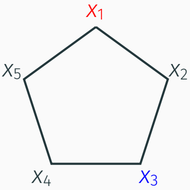
- Zero-error encoding of $N \Leftrightarrow$ Independent set of G_N

Zero-error Communication through Classical Channels

Classical communication



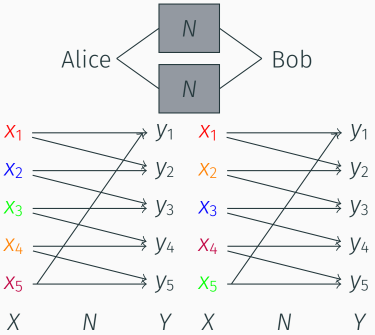
Confusability graph

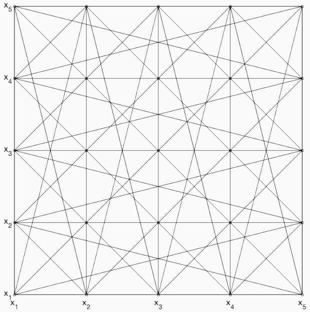


$x_i \sim x_j$ if $\exists y \in Y$, s.t. $N(y|x_i)N(y|x_j) > 0$.

- Zero-error encoding of $N \Leftrightarrow$ Independent set of G_N
- Maximum # zero-error messages send through N: $\alpha(G_N)$

Zero-error Communication through Classical Channels



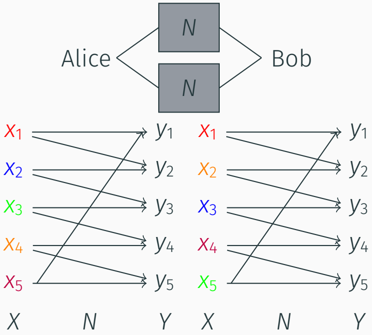


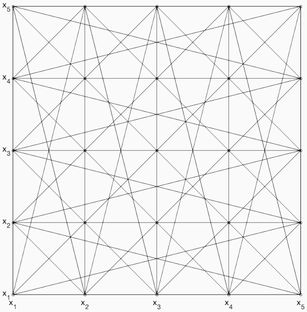
$\{\{(g,h),(g',h')\} \in E(G \boxtimes H)$ if

- $g = g'$, $\{h,h'\} \in E(H)$ or
- $\{g,g'\} \in E(G)$, $h = h'$ or
- $\{g,g'\} \in E(G)$, $\{h,h'\} \in E(H)$

- Zero-error encoding of $N \Leftrightarrow$ Independent set of G_N
- Maximum # zero-error messages send through N: $\alpha(G_N)$
- Block-code of length k through $N \Leftrightarrow$ confusability graph $G_N^{\boxtimes k}$

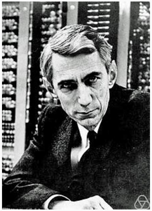
Zero-error Communication through Classical Channels



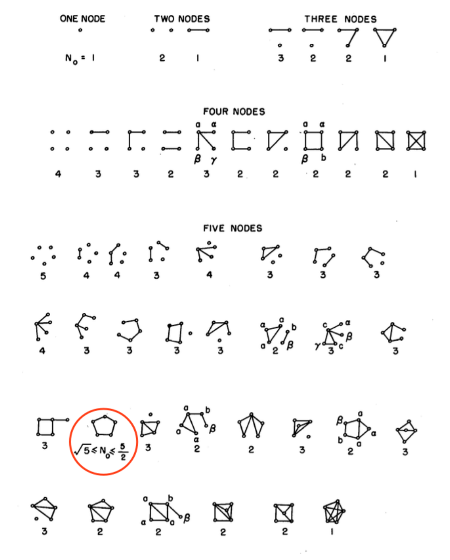


- Zero-error encoding of N ⇔ Independent set of G_N
- Maximum # zero-error messages send through N: $\alpha(G_N)$
- Block-code of length k through N ⇔ confusability graph $G_N^{\boxtimes k}$
- Shannon capacity (Shannon, 1956) of G_N (N):

$$\Theta(G_N) := \sup_k \sqrt[k]{\alpha(G_N^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(G_N^{\boxtimes k})}$$

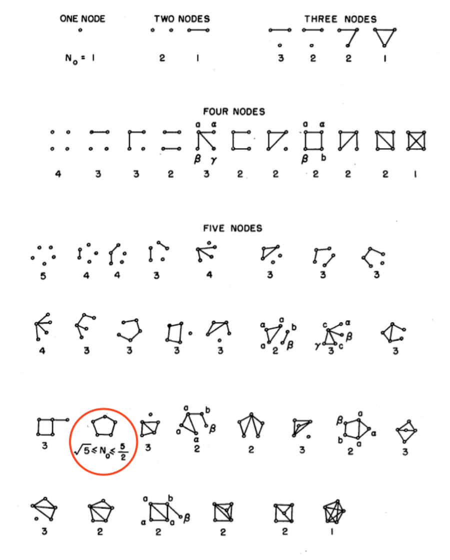


Compute the Shannon Capacity



(Shannon, 1956)

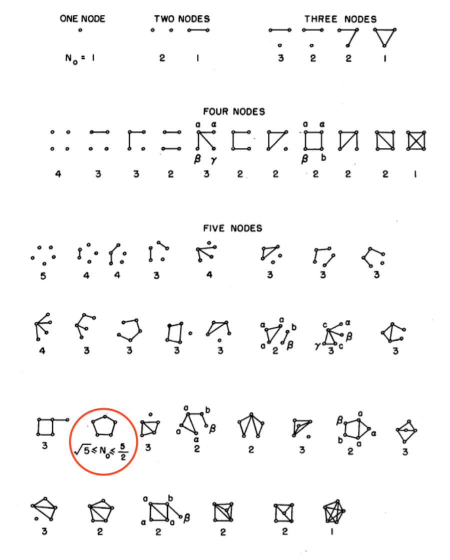
Compute the Shannon Capacity



(Shannon, 1956)

- $\Theta(G) \leq \overline{\chi}_f(G)$ (fractional clique cover number).

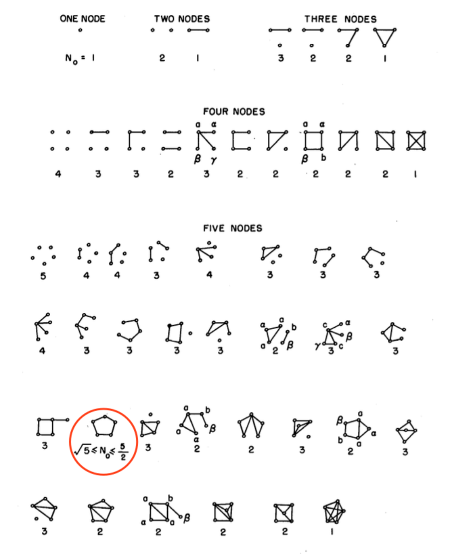
Compute the Shannon Capacity



(Shannon, 1956)

- $\Theta(G) \leq \overline{\chi}_f(G)$ (fractional clique cover number).
- Computing $\alpha(G)$ is NP-complete (Karp, 1972).

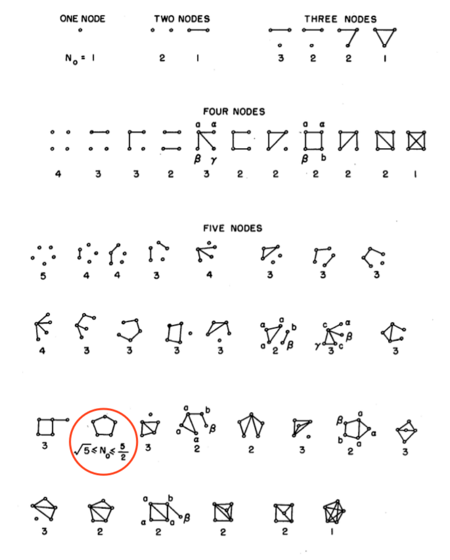
Compute the Shannon Capacity



(Shannon, 1956)

- $\Theta(G) \leq \overline{\chi}_f(G)$ (fractional clique cover number).
- Computing $\alpha(G)$ is NP-complete (Karp, 1972).
- $\exists$ G s.t. $\Theta(G) > \sqrt[k_0]{\alpha(G^{\boxtimes k_0})}$ for any $k_0 \in \mathbb{N}$ (Guo-Watanabe, 1990).

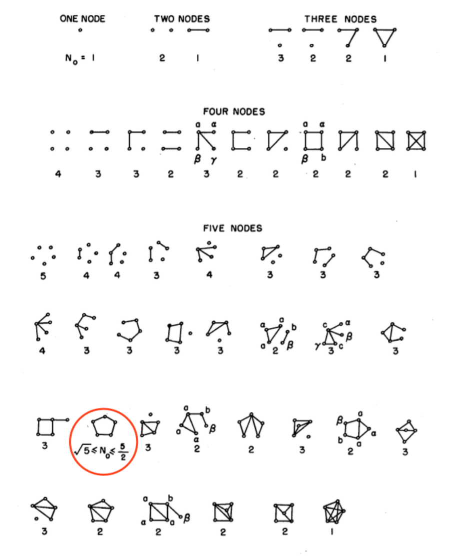
Compute the Shannon Capacity



(Shannon, 1956)

- $\Theta(G) \leq \overline{\chi}_f(G)$ (fractional clique cover number).
- Computing $\alpha(G)$ is NP-complete (Karp, 1972).
- $\exists$ G s.t. $\Theta(G) > \sqrt[k_0]{\alpha(G^{\boxtimes k_0})}$ for any $k_0 \in \mathbb{N}$ (Guo-Watanabe, 1990).
- Lower bounds: By construction. $\sqrt[5]{367} \leq \Theta(C_7)$ (Polak-Schrijver, 2019)

Compute the Shannon Capacity



(Shannon, 1956)

- $\Theta(G) \le \overline{\chi}_f(G)$ (fractional clique cover number).
- Computing $\alpha(G)$ is NP-complete (Karp, 1972).
- $\exists\, G$ s.t. $\Theta(G) > \sqrt[k_0]{\alpha(G^{\boxtimes k_0})}$ for any $k_0 \in \mathbb{N}$ (Guo-Watanabe, 1990).
- Lower bounds: By construction. $\sqrt[5]{367} \le \Theta(C_7)$ (Polak-Schrijver, 2019)
- Upper bounds: *Submultiplicative* upper bounds on $\alpha(G)$. $(f(G \boxtimes H) \le f(G)f(H))$

Lovász Theta Function (Lovász, submitted in Feb. 1978)

Orthogonal representation of $G = ([n], E)$:
$\Psi : [n] \to \mathbb{C}^d$, $\Psi(i) \mapsto |\psi_i\rangle$ s.t. $\langle\psi_i|\psi_j\rangle = 0$ if $\{i, j\} \notin E$.

Lovász Theta Function (Lovász, submitted in Feb. 1978)

Orthogonal representation of $G = ([n], E)$:
$\Psi : [n] \to \mathbb{C}^d$, $\Psi(i) \mapsto |\psi_i\rangle$ s.t. $\langle\psi_i|\psi_j\rangle = 0$ if $\{i,j\} \notin E$.

$$\vartheta(G) = \min_{\text{O.R. } \Psi} \min_{|\varphi\rangle \in \mathbb{C}^d} \max_{i \in [n]} |\langle\varphi|\psi_i\rangle|^{-2}$$

$$= \max\{\|I + T\| : T_{i,j} = 0 \text{ if } \{i,j\} \in E \text{ or } i = j,\ I + T \geq 0\}$$

Lovász Theta Function (Lovász, submitted in Feb. 1978)

Orthogonal representation of $G = ([n], E)$:
$\Psi : [n] \to \mathbb{C}^d$, $\Psi(i) \mapsto |\psi_i\rangle$ s.t. $\langle \psi_i | \psi_j \rangle = 0$ if $\{i, j\} \notin E$.

$$\vartheta(G) = \min_{\text{O.R. } \Psi} \min_{|\varphi\rangle \in \mathbb{C}^d} \max_{i \in [n]} |\langle \varphi | \psi_i \rangle|^{-2}$$

$$= \max\{\|I + T\| : T_{i,j} = 0 \text{ if } \{i, j\} \in E \text{ or } i = j,\ I + T \geq 0\}$$

$$\alpha(G) \leq \Theta(G) \leq \vartheta(G) \leq \overline{\xi}(G) \leq \overline{\chi}_f(G)$$

Orthogonal rank: $\overline{\xi}(G) = \min\{d : \exists \text{ O.R. } \Psi : [n] \to \mathbb{C}^d\}$

Lovász Theta Function (Lovász, submitted in Feb. 1978)

Orthogonal representation of $G = ([n], E)$:
$\Psi : [n] \to \mathbb{C}^d$, $\Psi(i) \mapsto |\psi_i\rangle$ s.t. $\langle \psi_i | \psi_j \rangle = 0$ if $\{i, j\} \notin E$.

$$\vartheta(G) = \min_{\text{O.R. } \Psi} \min_{|\varphi\rangle \in \mathbb{C}^d} \max_{i \in [n]} |\langle \varphi | \psi_i \rangle|^{-2}$$
$$= \max\{\|I + T\| : T_{i,j} = 0 \text{ if } \{i, j\} \in E \text{ or } i = j,\ I + T \succeq 0\}$$

$$\alpha(G) \leq \Theta(G) \leq \vartheta(G) \leq \overline{\xi}(G) \leq \overline{\chi}_f(G)$$

Orthogonal rank: $\overline{\xi}(G) = \min\{d : \exists \text{ O.R. } \Psi : [n] \to \mathbb{C}^d\}$

- $\sqrt{5} = \sqrt{\alpha(C_5^{\boxtimes 2})} \leq \Theta(C_5) \leq \vartheta(C_5) = \sqrt{5}$.
- $\Theta(G) = \vartheta(G)$ if G is a perfect graph.

Lovász Theta Function (Lovász, submitted in Feb. 1978)

Orthogonal representation of $G = ([n], E)$:
$\Psi : [n] \to \mathbb{C}^d$, $\Psi(i) \mapsto |\psi_i\rangle$ s.t. $\langle\psi_i|\psi_j\rangle = 0$ if $\{i,j\} \notin E$.

$$\vartheta(G) = \min_{\text{O.R. }\Psi} \min_{|\varphi\rangle\in\mathbb{C}^d} \max_{i\in[n]} |\langle\varphi|\psi_i\rangle|^{-2}$$

$$= \max\{\|I+T\| : T_{i,j} = 0 \text{ if } \{i,j\}\in E \text{ or } i=j,\ I+T \geq 0\}$$

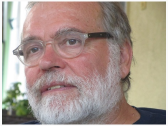
$$\alpha(G) \leq \Theta(G) \leq \vartheta(G) \leq \overline{\xi}(G) \leq \overline{\chi}_f(G)$$

Orthogonal rank: $\overline{\xi}(G) = \min\{d :\ \exists \text{ O.R. } \Psi : [n] \to \mathbb{C}^d\}$

- $\sqrt{5} = \sqrt{\alpha(C_5^{\boxtimes 2})} \leq \Theta(C_5) \leq \vartheta(C_5) = \sqrt{5}$.
- $\Theta(G) = \vartheta(G)$ if G is a perfect graph.
- Problem: Is $\Theta(G) = \vartheta(G)$ for any G?

The Haemers Bound (Haemers, submitted in Apr. 1978)

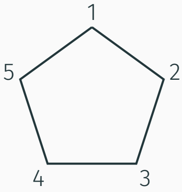
A matrix $A \in M_n(\mathbb{C})$ fits the graph $G = ([n], E)$:
$A_{i,i} = 1$ for $i \in [n]$ and $A_{i,j} = 0$ if $\{i, j\} \notin E$.



The Haemers Bound (Haemers, submitted in Apr. 1978)

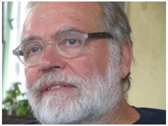
A matrix $A \in M_n(\mathbb{C})$ fits the graph $G = ([n], E)$:
$A_{i,i} = 1$ for $i \in [n]$ and $A_{i,j} = 0$ if $\{i, j\} \notin E$.

$$\alpha(G) \leq \mathrm{rank}\,(A),\ \alpha(G^{\boxtimes k}) \leq \mathrm{rank}\,(A^{\otimes k}) = \mathrm{rank}\,(A)^k$$



$G = C_5$

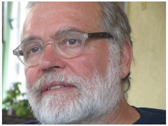
$$A = \begin{bmatrix} 1 & 5 & 0 & 0 & 2.4 \\ 0 & 1 & 1.7 + 3i & 0 & 0 \\ 0 & \pi & 1 & 0 & 0 \\ 0 & 0 & 1.1 + 2.2i & 1 & 20 \\ 7 & 0 & 0 & \pi/5 & 1 \end{bmatrix}$$



The Haemers Bound (Haemers, submitted in Apr. 1978)

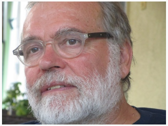
A matrix $A \in M_n(\mathbb{C})$ fits the graph $G = ([n], E)$:
$A_{i,i} = 1$ for $i \in [n]$ and $A_{i,j} = 0$ if$\{i, j\} \notin E$.

$\alpha(G) \leq \mathrm{rank}\,(A)$, $\alpha(G^{\boxtimes k}) \leq \mathrm{rank}\,(A^{\otimes k}) = \mathrm{rank}\,(A)^k$



$$\mathcal{H}(G) = \min\{\mathrm{rank}\,(A) : \ A \text{ fits } G\}$$
$$= \min\{\mathrm{rank}\,(I + B) : \ \ B_{i,j} = 0 \text{ if } \{i, j\} \notin E \text{ or } i = j\}$$

The Haemers Bound (Haemers, submitted in Apr. 1978)



A matrix $A \in M_n(\mathbb{C})$ fits the graph $G = ([n], E)$:
$A_{i,i} = 1$ for $i \in [n]$ and $A_{i,j} = 0$ if$\{i, j\} \notin E$.

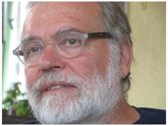
$\alpha(G) \leq \mathrm{rank}\,(A)$, $\alpha(G^{\boxtimes k}) \leq \mathrm{rank}\,(A^{\otimes k}) = \mathrm{rank}\,(A)^k$

$$\begin{aligned}\mathcal{H}(G) &= \min\{\mathrm{rank}\,(A) : \ A \text{ fits } G\} \\ &= \min\{\mathrm{rank}\,(I + B) : \ \ B_{i,j} = 0 \text{ if } \{i,j\} \notin E \text{ or } i = j\}\end{aligned}$$

$$\alpha(G) \leq \Theta(G) \leq \mathcal{H}(G) \leq \overline{\xi}(G) \leq \overline{\chi}_f(G)$$

$\overline{\xi}(G) = \min\{\mathrm{rank}\,(A) : \ A \text{ fits } G, \ A \geq 0\}$ (Peeters, 1996)

The Haemers Bound (Haemers, submitted in Apr. 1978)



A matrix $A \in M_n(\mathbb{C})$ fits the graph $G = ([n], E)$:
$A_{i,i} = 1$ for $i \in [n]$ and $A_{i,j} = 0$ if $\{i, j\} \notin E$.

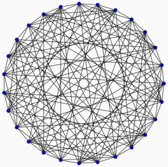
$\alpha(G) \leq \mathrm{rank}\,(A)$, $\alpha(G^{\boxtimes k}) \leq \mathrm{rank}\,(A^{\otimes k}) = \mathrm{rank}\,(A)^k$

$$\begin{aligned}\mathcal{H}(G) &= \min\{\mathrm{rank}\,(A) : \ A \text{ fits } G\} \\ &= \min\{\mathrm{rank}\,(I + B) : \ \ B_{i,j} = 0 \text{ if } \{i, j\} \notin E \text{ or } i = j\}\end{aligned}$$

$$\alpha(G) \leq \Theta(G) \leq \mathcal{H}(G) \leq \overline{\xi}(G) \leq \overline{\chi}_f(G)$$

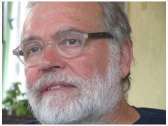
$\overline{\xi}(G) = \min\{\mathrm{rank}\,(A) : \ A \text{ fits } G, \ A \geq 0\}$ (Peeters, 1996)

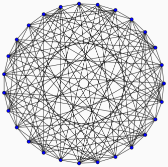
- $\mathcal{H}(G) < \vartheta(G) \Rightarrow \Theta(G) \neq \vartheta(G)$, further $\Theta(G)\Theta(\overline{G}) < \Theta(G \boxtimes \overline{G})$.



The complement of the Schläfli graph.

The Haemers Bound (Haemers, submitted in Apr. 1978)





The complement of the Schläfli graph.

A matrix $A \in M_n(\mathbb{C})$ fits the graph $G = ([n], E)$:
$A_{i,i} = 1$ for $i \in [n]$ and $A_{i,j} = 0$ if $\{i, j\} \notin E$.

$$\alpha(G) \leq \mathrm{rank}(A),\ \alpha(G^{\boxtimes k}) \leq \mathrm{rank}(A^{\otimes k}) = \mathrm{rank}(A)^k$$

$$\begin{aligned} \mathcal{H}(G) &= \min\{\mathrm{rank}(A) :\ A \text{ fits } G\} \\ &= \min\{\mathrm{rank}(I + B) :\ \ B_{i,j} = 0 \text{ if } \{i, j\} \notin E \text{ or } i = j\} \end{aligned}$$

$$\alpha(G) \leq \Theta(G) \leq \mathcal{H}(G) \leq \overline{\xi}(G) \leq \overline{\chi}_f(G)$$

$\overline{\xi}(G) = \min\{\mathrm{rank}(A) :\ A \text{ fits } G,\ A \geq 0\}$ (Peeters, 1996)

- $\mathcal{H}(G) < \vartheta(G) \Rightarrow \Theta(G) \neq \vartheta(G)$, further $\Theta(G)\Theta(\overline{G}) < \Theta(G \boxtimes \overline{G})$.
- Also satisfy $\Theta(G) + \Theta(\overline{G}) < \Theta(G \sqcup \overline{G})$. (Alon, 1998)
- $\mathcal{H}(G)$ can be defined over any field. Used to separate Shannon capacity and entangled Shannon capacity. (LMMOR, 2012, BBG, 2013)

Zero-error communication in a quantum world and a generalization of the Haemers bound

Zero-error Communication through Quantum Channels

Quantum Channel: $\Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^{\dagger} \ \forall \rho \in M_n$ satisfying $\sum_{k=1}^{m} E_k^{\dagger} E_k = I_n$

Zero-error Communication through Quantum Channels

Quantum Channel: $\Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \ \forall \rho \in M_n$ satisfying $\sum_{k=1}^{m} E_k^\dagger E_k = I_n$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

Zero-error Communication through Quantum Channels

$$\text{Quantum Channel: } \Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \ \forall \rho \in M_n \text{ satisfying } \sum_{k=1}^{m} E_k^\dagger E_k = I_n$$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \text{ and } \Phi(|\psi_j\rangle\langle\psi_j|) \text{ are nonconfusable } \Leftrightarrow \Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|)$$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle|^2 = 0$$

Zero-error Communication through Quantum Channels

$$\text{Quantum Channel: } \Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \; \forall \rho \in M_n \text{ satisfying } \sum_{k=1}^{m} E_k^\dagger E_k = I_n$$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

$\Phi(|\psi_i\rangle\langle\psi_i|)$ and $\Phi(|\psi_j\rangle\langle\psi_j|)$ are nonconfusable $\Leftrightarrow$ $\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle = 0 \; \forall \, k, k' = 1, \ldots, m$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle|^2 = 0$$

Zero-error Communication through Quantum Channels

$$\text{Quantum Channel: } \Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \; \forall \rho \in M_n \text{ satisfying } \sum_{k=1}^{m} E_k^\dagger E_k = I_n$$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

$\Phi(|\psi_i\rangle\langle\psi_i|)$ and $\Phi(|\psi_j\rangle\langle\psi_j|)$ are nonconfusable $\Leftrightarrow$ $\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle = 0 \; \forall \, k, k' = 1, \ldots, m$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle|^2 = 0$$

Noncommutative (confusability) graph (Duan-Severini-Winter, 2013):

$$S_\Phi = \mathrm{span}\{E_k^\dagger E_{k'} \; : \; k, k' = 1, \ldots, m\} \subseteq M_n$$

Zero-error Communication through Quantum Channels

$$\text{Quantum Channel: } \Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \; \forall \rho \in M_n \text{ satisfying } \sum_{k=1}^{m} E_k^\dagger E_k = I_n$$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

$\Phi(|\psi_i\rangle\langle\psi_i|)$ and $\Phi(|\psi_j\rangle\langle\psi_j|)$ are nonconfusable $\Leftrightarrow$ $\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle = 0 \; \forall \; k, k' = 1, \ldots, m$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle|^2 = 0$$

Noncommutative (confusability) graph (Duan-Severini-Winter, 2013):

$$S_\Phi = \mathrm{span}\{E_k^\dagger E_{k'} \; : \; k, k' = 1, \ldots, m\} \subseteq M_n$$

Independence number (Maximum # zero-error messages send through Φ):

Zero-error Communication through Quantum Channels

Quantum Channel: $\Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \ \forall \rho \in M_n$ satisfying $\sum_{k=1}^{m} E_k^\dagger E_k = I_n$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

$\Phi(|\psi_i\rangle\langle\psi_i|)$ and $\Phi(|\psi_j\rangle\langle\psi_j|)$ are nonconfusable $\Leftrightarrow \langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle = 0 \ \forall\, k, k' = 1, \ldots, m$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle|^2 = 0$$

Noncommutative (confusability) graph (Duan-Severini-Winter, 2013):

$$S_\Phi = \mathrm{span}\{E_k^\dagger E_{k'} : k, k' = 1, \ldots, m\} \subseteq M_n$$

Independence number (Maximum # zero-error messages send through Φ):

$$\alpha(S_\Phi) = \max\{\ell : \exists \underbrace{|\psi_1\rangle, \ldots, |\psi_\ell\rangle}_{\text{“vertices”}}, \underbrace{\langle\psi_i|A|\psi_j\rangle = 0}_{\text{“nonadjacency”}} \ \forall\, i \neq j \ \& \ \underbrace{A \in S_\Phi}_{\text{“edges”}}\}.$$

Zero-error Communication through Quantum Channels

Quantum Channel: $\Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^\dagger \ \forall \rho \in M_n$ satisfying $\sum_{k=1}^{m} E_k^\dagger E_k = I_n$

Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

$\Phi(|\psi_i\rangle\langle\psi_i|)$ and $\Phi(|\psi_j\rangle\langle\psi_j|)$ are nonconfusable $\Leftrightarrow$ $\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle = 0 \ \forall\, k, k' = 1, \ldots, m$

$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i|E_k^\dagger E_{k'}|\psi_j\rangle|^2 = 0$$

Noncommutative (confusability) graph (Duan-Severini-Winter, 2013):

$$S_\Phi = \mathrm{span}\{E_k^\dagger E_{k'} : \ k, k' = 1, \ldots, m\} \subseteq M_n$$

Independence number (Maximum # zero-error messages send through Φ):

$$\alpha(S_\Phi) = \max\{\ell : \ \exists \underbrace{|\psi_1\rangle, \ldots, |\psi_\ell\rangle}_{\text{“vertices”}}, \ \underbrace{\langle\psi_i|A|\psi_j\rangle = 0}_{\text{“nonadjacency”}} \ \forall\, i \neq j \ \& \ \underbrace{A \in S_\Phi}_{\text{“edges”}}\}.$$

The Shannon capacity of S_Φ: $\Theta(S_\Phi) := \sup_k \sqrt[k]{\alpha(S_\Phi^{\otimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(S_\Phi^{\otimes k})}$

Compute Noncommutative Shannon Capacity

Confusability graph $G = ([n], E) \Rightarrow$ Noncommutative graph

$$S_G = \mathrm{span}\{|i\rangle\langle j| : \ i = j \in [n], \text{ or } \{i, j\} \in E\}.$$

Compute Noncommutative Shannon Capacity

Confusability graph $G = ([n], E) \Rightarrow$ Noncommutative graph

$$S_G = \mathrm{span}\{|i\rangle\langle j| : \ i = j \in [n], \text{ or } \{i, j\} \in E\}.$$

(DSW13): $\alpha(G) = \alpha(S_G)$, $\Theta(G) = \Theta(S_G)$

Compute Noncommutative Shannon Capacity

Confusability graph $G = ([n], E) \Rightarrow$ Noncommutative graph

$$S_G = \mathrm{span}\{|i\rangle\langle j| : \; i = j \in [n], \text{ or } \{i,j\} \in E\}.$$

(DSW13): $\alpha(G) = \alpha(S_G)$, $\Theta(G) = \Theta(S_G)$

Decide whether $\alpha(S) \geq k$ is QMA-hard. (Beigi, Shor, 2008)

Upper bound noncommutative Shannon capacity:

- Start from a graph-theoretic submultiplicative upper bound $f(G)$ on $\alpha(G)$.
- "Extend" f to f_q acting on noncommutative graphs, s.t. $f_q(S_G) = f(G)$.
- Prove $f_q(S)$ upper bounds $\alpha(S)$, and is submultiplicative (w.r.t. $\otimes$).

Noncomm. Lovász Theta Function (DSW13)

The Lovász theta function:

$$\begin{aligned}\vartheta(G) &= \max\{||I+T|| : I+T \geq 0,\ T_{i,j}=0 \text{ if } \{i,j\} \in E \text{ or } i=j\} \\ &= \max\{||I+T|| : I+T \geq 0, T \in S_G^{\perp}\}\end{aligned}$$

Noncomm. Lovász Theta Function (DSW13)

The Lovász theta function:

$$\vartheta(G) = \max\{||I+T|| : I+T \geq 0,\ T_{i,j} = 0 \text{ if } \{i,j\} \in E \text{ or } i = j\}$$
$$= \max\{||I+T|| : I+T \geq 0, T \in S_G^{\perp}\}$$

A naïve generalization: $\vartheta_q(S) = \max\{||I+T|| : I+T \geq 0, T \in S^{\perp}\}$.

Noncomm. Lovász Theta Function (DSW13)

The Lovász theta function:

$$\begin{aligned}\vartheta(G) &= \max\{||I+T|| : I+T \geq 0,\ T_{i,j} = 0 \text{ if } \{i,j\} \in E \text{ or } i=j\} \\ &= \max\{||I+T|| : I+T \geq 0, T \in S_G^{\perp}\}\end{aligned}$$

A naïve generalization: $\vartheta_q(S) = \max\{||I+T|| : I+T \geq 0, T \in S^{\perp}\}$.

- $\vartheta_q(S_G) = \vartheta(G)$ for any graph G.
- $\vartheta_q(S) \geq \alpha(S)$ for any noncommutative graph S.
- ϑ_q is only **supermultiplicative**.

Noncomm. Lovász Theta Function (DSW13)

The Lovász theta function:

$$\vartheta(G) = \max\{||I+T|| : I+T \geq 0,\ T_{i,j} = 0 \text{ if } \{i,j\} \in E \text{ or } i = j\}$$
$$= \max\{||I+T|| : I+T \geq 0, T \in S_G^{\perp}\}$$

A naïve generalization: $\vartheta_q(S) = \max\{||I+T|| : I+T \geq 0, T \in S^{\perp}\}$.

- $\vartheta_q(S_G) = \vartheta(G)$ for any graph G.
- $\vartheta_q(S) \geq \alpha(S)$ for any noncommutative graph S.
- ϑ_q is only **supermultiplicative**.

Need a norm completion:

$$\tilde{\vartheta}_q(S) = \sup_m \vartheta_q(S \otimes M_m) = \sup_m \max\{||I+T|| : I+T \geq 0, T \in S^{\perp} \otimes M_m\}$$

- $\tilde{\vartheta}_q(S_G) = \vartheta(G)$ for any graph G.
- $\tilde{\vartheta}_q(S) \geq \alpha(S)$ for any noncommutative graph S.
- $\tilde{\vartheta}_q(S)$ can be computed by semidefinite programming.
- $\tilde{\vartheta}_q(S \otimes T) = \tilde{\vartheta}_q(S)\tilde{\vartheta}_q(T)$, $\tilde{\vartheta}_q(S \oplus T) = \tilde{\vartheta}_q(S) + \tilde{\vartheta}_q(T)$.
- $\tilde{\vartheta}_q(S) \geq \Theta(S)$ for any noncommutative graph S.

From the Orthogonal Rank to the Haemers Bound

$$\begin{aligned}\mathcal{H}(G) &= \min\{\mathrm{rank}\,(I+B) : \; B_{i,j} = 0 \text{ if } \{i,j\} \notin E \text{ or } i = j\} \\ &= \min\{\mathrm{rank}\,(I+B) : \; B \in S_G^{\perp}\}\end{aligned}$$

It is not clear how to get $\mathcal{H}_q$.

From the Orthogonal Rank to the Haemers Bound

$$\mathcal{H}(G) = \min\{\mathrm{rank}\,(I+B) : \quad B_{i,j} = 0 \text{ if } \{i,j\} \notin E \text{ or } i = j\}$$
$$= \min\{\mathrm{rank}\,(I+B) : \quad B \in S_G^{\perp}\}$$

It is not clear how to get $\mathcal{H}_q$.

Recall:

- $\overline{\xi}(G)$: Min. dim. of an O.R. $\Leftrightarrow$ Haemers bound + positive semidefinite matrices.
- O. R. $\Psi : [n] \to \mathbb{C}^d \Leftrightarrow$ classical-quantum channel $\Psi(\rho) = \sum_{i=1}^{n} |\psi_i\rangle\langle i|\rho|i\rangle\langle\psi_i|$
- Note $S_\Psi = \mathrm{span}\{|i\rangle\langle j| : \langle\psi_i|\psi_j\rangle \neq 0\} \subseteq S_G$.
- $\overline{\xi}_q(S) = \min\{d : \exists$ Q. C. $\Phi : M_n \to M_d$, such that $S_\Phi \subseteq S\}$. (Stahkle, 2016)

From the Orthogonal Rank to the Haemers Bound

$$\mathcal{H}(G) = \min\{\mathrm{rank}\,(I+B) : \quad B_{i,j} = 0 \text{ if } \{i,j\} \notin E \text{ or } i = j\}$$
$$= \min\{\mathrm{rank}\,(I+B) : \quad B \in S_G^{\perp}\}$$

It is not clear how to get $\mathcal{H}_q$.

Recall:

- $\overline{\xi}(G)$: Min. dim. of an O.R. $\Leftrightarrow$ Haemers bound + positive semidefinite matrices.
- O. R. $\Psi : [n] \to \mathbb{C}^d \Leftrightarrow$ classical-quantum channel $\Psi(\rho) = \sum_{i=1}^{n} |\psi_i\rangle\langle i|\rho|i\rangle\langle\psi_i|$
- Note $S_\Psi = \mathrm{span}\{|i\rangle\langle j| : \langle\psi_i|\psi_j\rangle \neq 0\} \subseteq S_G$.
- $\overline{\xi}_q(S) = \min\{d : \exists$ Q. C. $\Phi : M_n \to M_d$, such that $S_\Phi \subseteq S\}$. (Stahkle, 2016)

(Levene-Paulsen-Todorov, 2018):

$$\overline{\xi}_q(S) = \min\{\mathrm{rank}\,(B) : \; m \in \mathbb{N},\; B \in M_m(S),\; \sum_{i=1}^{m} B_{i,i} = I_n,\; B \geq 0\}$$

The Haemers Bound of Noncommutative Graphs

Haemers bound of noncommutative graph $S \subseteq M_n$ (Gribling, Li, 2020):

$$\mathcal{H}_q(S) = \min\{\mathrm{rank}\,(B) :\; m \in \mathbb{N},\; B \in M_m(S),\; \sum_{i=1}^{m} B_{i,i} = I_n\}$$

The Haemers Bound of Noncommutative Graphs

Haemers bound of noncommutative graph $S \subseteq M_n$ (Gribling, Li, 2020):

$$\mathcal{H}_q(S) = \min\{\mathrm{rank}\,(B) : \ m \in \mathbb{N},\ B \in M_m(S),\ \sum_{i=1}^{m} B_{i,i} = I_n\}$$

- $\mathcal{H}_q(S_G) = \mathcal{H}(G)$ for any graph G.
- $\Theta(S) \leq \mathcal{H}_q(S)$ for any noncommutative graph S.
  - $\alpha(S) \leq \mathcal{H}_q(S)$ for any noncommutative graph S.
  - $\mathcal{H}_q(S \otimes T) \leq \mathcal{H}_q(S)\mathcal{H}_q(T)$ for any noncommutative graphs S and T.

The Haemers Bound of Noncommutative Graphs

Haemers bound of noncommutative graph $S \subseteq M_n$ (Gribling, Li, 2020):

$$\mathcal{H}_q(S) = \min\{\mathrm{rank}\,(B) : \ m \in \mathbb{N},\ B \in M_m(S),\ \sum_{i=1}^{m} B_{i,i} = I_n\}$$

- $\mathcal{H}_q(S_G) = \mathcal{H}(G)$ for any graph G.
- $\Theta(S) \leq \mathcal{H}_q(S)$ for any noncommutative graph S.
  - $\alpha(S) \leq \mathcal{H}_q(S)$ for any noncommutative graph S.
  - $\mathcal{H}_q(S \otimes T) \leq \mathcal{H}_q(S)\mathcal{H}_q(T)$ for any noncommutative graphs S and T.
- Operational interpretation:

$$\mathcal{H}_q(S) = \min\{d : \exists \text{ trace-preserving map } \Phi : M_n \to M_d, \text{s.t. } S_\Phi \subseteq S\}$$

The Haemers Bound of Noncommutative Graphs

Haemers bound of noncommutative graph $S \subseteq M_n$ (Gribling, Li, 2020):

$$\mathcal{H}_q(S) = \min\{\mathrm{rank}\,(B) : \; m \in \mathbb{N}, \; B \in M_m(S), \; \sum_{i=1}^{m} B_{i,i} = I_n\}$$

- $\mathcal{H}_q(S_G) = \mathcal{H}(G)$ for any graph G.
- $\Theta(S) \leq \mathcal{H}_q(S)$ for any noncommutative graph S.
  - $\alpha(S) \leq \mathcal{H}_q(S)$ for any noncommutative graph S.
  - $\mathcal{H}_q(S \otimes T) \leq \mathcal{H}_q(S)\mathcal{H}_q(T)$ for any noncommutative graphs S and T.
- Operational interpretation:

  $$\mathcal{H}_q(S) = \min\{d : \exists \text{ trace-preserving map } \Phi : M_n \to M_d, \text{s.t. } S_\Phi \subseteq S\}$$

- $\mathcal{H}_q(S)$ is computable:
  - $\mathcal{H}_q(S)$ can be achieved with $m \leq n^4$
  - Compute $\mathcal{H}_q(S)$ using Hilbert's Nullstellensatz.

Summary

- A definition of the Haemers bound of noncommutative graphs.
- Upper bound the Shannon (resp. zero-error) capacity of noncommutative graphs (resp. quantum channels).
- Operational meaning and computability of the Haemers bound.
- Can be better than any other previous bounds.
- Work in progress: Developing a mathematical theory of noncommutative graphs and its connections to other problems in zero-error quantum information theory.

Photo taken after Sander’s Ph.D. defence.